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广义 KDV 方程的显示行波解

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摘要: 非线性演化方程, 特别是广义 KDV 方程因其丰富的数学物理内含而备受人们关注, 其精确解的研究在理论和应用上都有重要的意义. 求出了广义 KDV 方程的显示精确解, 同时给出了解成立的条件, 其求解方法也适用于求解其它非线性演化方程.

关键词: 非线性演化方程; 行波解; 精确解; 孤立波
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0 引言

最近几十年, 在物理学、流体动力学、生物、化学等许多研究领域中发现大量非线性现象, 这些现象均可用非线性演化方程的数学模型加以描述, 方程及其精确解的研究受到广泛重视和深入研究[1], 其解的多样性反映出物质世界各种形式的时空结构. 但囿于数学上的复杂性, 求非线性演化方程的显示精确解极度困难. 若是求其行波解, 则可将非线性偏微分方程化为非线性常微分方程, 从而设法求出非线性演化方程精确行波解.

考虑如下形式的广义 KDV 方程

$$u_t+\beta u^{\alpha}u_x+\gamma u^{\tau}u_{xxx}=0, \tag{1}$$

其中, α、β、γ 和 τ 均为实数.

当 $\tau=0$, $\alpha=1$时, 方程(1)还原为经典的 KDV 方程

$$u_t+\beta u_x+\gamma u_{xxx}=0; \tag{2}$$

当 $\tau=0$, $\alpha=2$ 时, 方程(1)还原为 MKDV 方程, 可作为一维非线性晶格传播波的模型

$$u_t+\beta u^2u_x+\gamma u_{xxx}=0; \tag{3}$$

当 $\tau=0$, $\alpha=3$ 时, 用来描述在不同电荷和温度下等离子体中的离子-声子波; 当 $\tau=0$, $\alpha=1/2$, $\beta=1$ 时, H. Schamel[2] 用来描述冷等离子体中的离子-声子波.

文[3] 对许多类型的 KDV 方程的精确解作了总结, 目前尚未见到对广义 KDV 方程的讨论. 本文采用直接综合假设方法求出方程(1)的显示精确解.

1 广义 KDV 方程的显示行波解

考虑非线性演化方程(1)的显示行波解, 可令 $u=u(\xi)$, $\xi=x-ct$, 其中 c 为波速, 代入方程(1)得非线性常微方程

$$-cu_{\xi}+\beta u^{\alpha}u_{\xi}+\gamma u^{\tau}u_{\xi\xi\xi}=0. \tag{4}$$

通过下面 3 种综合假设, 可求出(1)的行波解.

(1) 设方程(1)的解满足

$$u'=-vbu\sqrt{1-\left(\frac{u}{a}\right)^{2/v}}, \tag{5}$$

通过积分得

$$u(\xi)=a\,\mathrm{sech}^{v}(b\xi+c_0), \tag{6}$$

这里 a, b, v 为实数(大小待定). 微分后得

$$\left.\begin{aligned} u''&=v^2b^2u[1-(1+1/v)(u/a)^{2/v}]=\\ &-\frac{vb}{\sqrt{1-(u/a)^{2/v}}}[1-(1+1/v)(u/a)^{2/v}]\,u',\\ u'''&=v^2b^2\left[1-\frac{(v+1)(v+2)}{v^2}(u/a)^{2/v}\right]u'. \end{aligned}\right\} \tag{7}$$

将以上结果代入方程(4)得

$$-c+\beta u^{\alpha}+\gamma u^{\tau}v^2b^2\left[1-\frac{(v+1)(v+2)}{v^2}(u/a)^{2/v}\right]=0, \tag{8}$$

$$\left.\begin{aligned} &\alpha=\tau=-2/v,\\ &\beta+\gamma v^2b^2=0,\\ &-c-\frac{\gamma v^2b^2}{v^2}(v+1)(v+2)a^{-2/v}=0. \end{aligned}\right\} \tag{9}$$

解之得

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$$\left.\begin{aligned} b &= \pm\sqrt{\frac{\alpha^2\beta}{4\gamma}},\\ a &= \left[\frac{2c}{\beta(\alpha-1)(\alpha-2)}\right]^{1/\alpha}.\end{aligned}\right\}\tag{10}$$

(2) 设方程(1)的解满足

$$u' = vbu(a/u)^{1/v}[1-(u/a)^{2/v}],\tag{11}$$

积分后得

$$u(\xi) = a\tanh^v(b\xi + c_0),\tag{12}$$

$$\left.\begin{aligned} &u' = vb^2u[(v-1)(u/a)^{2/v} - 2v + \\ &(v+1)(u/a)^{2/v}] = b(a/u)^{1/v}\times \\ &[v-1-(v+1)(u/a)^{2/v}]u',\\ &u''' = b^2[(v-1)(v-2)(a/u)^{2/v} - \\ &-2v^2 + (v+1)(v+2)(\frac{u}{a})^{2/v}] = u'.\end{aligned}\right\}\tag{13}$$

将以上结果代入方程(4)得

$$-c + \beta u^\alpha + \gamma u^\tau b^2[(v-1)(v-2)(\frac{a}{u})^{2/v} - 2v^2 + (v+1)(v+2)(\frac{u}{a})^{2/v}] = 0.\tag{14}$$

上式成立分几种情况讨论:

(i) $v = 1$, $\alpha = \tau - 2/v$ 时,

$$\left.\begin{aligned} &-c + \gamma b^2(v+1)(v+2)a^{-2/v} = 0,\\ &\beta - \gamma b^2(2v^2) = 0.\end{aligned}\right\}\tag{15}$$

解之得

$$\left.\begin{aligned} a &= (\frac{c}{3\beta})^{-1/2},\\ b &= \pm\sqrt{\beta/2\gamma}.\end{aligned}\right\}\tag{16}$$

(ii) $v = 2$, $\alpha = \tau = -2/v$ 时,

$$\left.\begin{aligned} a &= 3\beta/2c,\\ b &= \pm\sqrt{\frac{\beta}{8\gamma}}.\end{aligned}\right\}\tag{17}$$

(iii) $\tau = 0$, $\alpha = 2/v$ ($v = 1$, 或 $v = 2$) 时,

$$\left.\begin{aligned} &-c - 2v^2\gamma b^2 = 0,\\ &\beta + \gamma b^2(v+1)(v+2)a^{-2/v} = 0,\\ &-c - 2v^2\gamma b^2 = 0,\\ &\beta = \gamma b^2(v+1)(v+2)a^{-2/v} = 0.\end{aligned}\right\}\tag{18}$$

解之得

$$\left.\begin{aligned} a &= (-\frac{3c}{\beta})^{1/2} \text{或} -\frac{3c}{2\beta},\\ b &= \pm\sqrt{\frac{-c}{2\gamma}} \text{或} \pm\sqrt{\frac{-c}{8\gamma}}.\end{aligned}\right\}\tag{19}$$

(3) 设方程(1)的解满足

$$u' = -vbu\sqrt{1+(u/a)^{2/v}},\tag{20}$$

积分后得

$$u(\xi) = a\,\mathrm{cosech}^v(b\xi + c_0).\tag{21}$$

微分后得

$$\left.\begin{aligned} &u'' = v^2b^2u[1+\frac{v+1}{v}(u/a)^{2/v}] = \\ &-\frac{vb}{\sqrt{1+(u/a)^{2/v}}}[1+(v+1)(u/a)^{2/v}/v]u',\\ &u''' = v^2b^2[1+(v+1)(v+2)(\frac{u}{a})^{2/v}/v^2]u'.\end{aligned}\right\}\tag{22}$$

将以上结果代入方程(5)得

$$-c + \beta u^\alpha + \gamma u^\tau v^2b^2\times [1+(v+1)(v+2)(\frac{u}{a})^{2/v}/v^2] = 0.\tag{23}$$

上式成立分几种情况讨论:

(i) $\alpha = \tau = -2/v$ 时,

$$\left.\begin{aligned} &-c + \gamma b^2v^2(v+1)(v+2)a^{-2/v}/v^2 = 0,\\ &\beta + \gamma b^2v^2 = 0.\end{aligned}\right\}\tag{24}$$

解之得

$$\left.\begin{aligned} a &= \left[\frac{-2c}{(\alpha-1)(\alpha-2)\beta}\right]^{1/\alpha},\\ b &= \pm\frac{\alpha}{2}\sqrt{-\beta/\gamma}.\end{aligned}\right\}\tag{25}$$

(ii) $\tau = 0$, $\alpha = 2/v$ 时,

$$\left.\begin{aligned} &-c + 2v^2\gamma b^2 = 0,\\ &\beta + \gamma b^2(v+1)(v+2)a^{-2/v} = 0.\end{aligned}\right\}\tag{26}$$

解之得

$$a = \left[\frac{-2\beta}{c(\alpha+1)(\alpha+2)}\right]^{-1/\alpha},$$

$$b = \pm\frac{\alpha}{2}\sqrt{\frac{c}{\gamma}}.\tag{27}$$

(4) 设方程(1)的解满足

$$u = vbu\sqrt{(a/u)^{1/v} - 1},\tag{28}$$

积分微分后得

$$\left.\begin{aligned} &u(\xi) = a\sin^v(b\xi + c_0),\\ &u'' = -v^2b^2u[1-(1-\frac{1}{v})(a/u)^{2/v}] = \\ &-\frac{vb}{\sqrt{(a/u)^{2/v}-1}}[1-(1-1/v)(a/u)^{2/v}]u',\\ &u''' = -v^2b^2[1-(v-1)(v-2)(\frac{a}{u})^{2/v}/v^2]u'.\end{aligned}\right\}\tag{29}$$

将以上结果代入方程(4)得

$$-c + \beta u^\alpha + \gamma u^\tau v^2b^2\times [1-(v-1)(v-2)(\frac{a}{u})^{2/v}/v^2] = 0.\tag{30}$$

上式成立须满足

$$\left.\begin{aligned} &\alpha = \tau = 2/v, \\ &-c + \gamma b^2 v^2 (v-1)(v-2) a^{2/v} = 0, \\ &\beta - \gamma b^2 v^2 = 0. \end{aligned}\right\} \quad (31)$$

解之得

$$\left.\begin{aligned} &a = \left[\frac{2c}{(\alpha-1)(\alpha-2)\beta}\right]^{1/\alpha}, \\ &b = \pm\frac{\alpha}{2}\sqrt{\beta/\gamma}, \\ &(\alpha \neq 1,\ \alpha \neq 2). \end{aligned}\right\} \quad (32)$$

2 结论

通过直接应用综合假设方法求出了广义 KDV 方程的显示行波精确解.这些解反映了各种形式的波结构,不但包含已有文献的一些特殊结果,还得到一些新的特解,并给出了解的条件,其求解分析方法可推广应用到其它非线性演化方程.

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The Exact Traveling Wave Solutions for Generalized KDV Equations

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Abstract: Nonlinear evolution equations especially generalized KDV equations have been studied intensively, because they contain abundant mathematic physics meaning. Study of their exact solutions is very important for both theory and applications. A series of traveling (or solitary) solutions of the KDV equations are studied in this paper and the conditions for the solutions aslo are investigated. Method used to solve the solutions can been used to analyse other nonlinear evolution equations.

Key words: Nonlinear evolution equation; Traveling wave; Exact solution; Solitary

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